Chapter IV Limit of Functions

Let $\emptyset \neq A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ throughout. Let

$$A^c := \{ c \in \mathbb{R} : V_\delta(c) \text{ intersects } A \setminus \{c\} \ \forall \delta > 0 \}$$
$$= \{ c \in \mathbb{R} : \forall \delta > 0 \ \exists\, a \in A \text{ s.t. } 0 < |a - c| < \delta \}$$
$$= \{ c \in \mathbb{R} : \forall n \in \mathbb{N}, \exists\, a_n \in A \setminus \{c\} \text{ s.t. } |a_n - c| < \tfrac{1}{n} \}$$
$$= \{ c \in \mathbb{R} : \exists \text{ a seq. } (a_n) \text{ in } A \setminus \{c\} \text{ s.t. } \lim a_n = c \}.$$

An element of A^c is called a cluster point (or accumulation point) w.r.t. A even though it may not be in A.

$c \in A^c$ iff $\operatorname{dist}(c, A \setminus \{c\}) := \inf \{ |c - a| : a \in A \setminus \{c\} \} = 0$

iff $\operatorname{dist}(c, A \setminus \{c\}) < \varepsilon \ \forall \varepsilon > 0$.

For example, if

$$A := (\sqrt{2}, 3) \cap \mathbb{Q}$$

then $A^c = [\sqrt{2}, 3]$. Indeed $A^c \supseteq [\sqrt{2}, 3]$ because if $\sqrt{2} \le x < 3$ and $\delta > 0$ then, with $\delta' = \min\{\delta, 3 - x\}$, one takes (by density of $\mathbb{Q}$) some rational c s.t. $x < c < x + \delta' \le 3$ so $c \in A$ and $0 < |x - c| < \delta' \le \delta$ implying that $x \in A^c$; if $\sqrt{2} < x \le 3$ then one shows similarly that $x \in A^c$.

To show $A^c \subseteq [\sqrt{2}, 3]$, you are asked to check that if $x < \sqrt{2}$ or $3 < x$ then $V_\delta(x)$ is disjoint from $A \setminus \{x\}$ (in fact disjoint from $[\sqrt{2}, 3]$) for some $\delta > 0$.

Below we attempt to define $\lim\limits_{\substack{x \to c \\ x \in A}} f(x) = \ell$

Definition of Limits. Let $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$, $L \in \mathbb{R}$. We say that $f(x)$ converges to L as x converges to c ($f(x) \to L$ as $x \to c$) if, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that

(*) $\quad |f(x) - L| < \varepsilon$ whenever $x \in A \setminus \{c\}$ with $0 < |x - c| < \delta$

Note. Two Cases:

Case 1: $c \notin A^c$ (i.e. c is isolated from A): $\exists\, \delta_0 > 0$ s.t. $V_{\delta_0}(c) \cap (A \setminus \{c\}) =$ empty

(The <u>punched</u> neighbourhood $V'_{\delta_0}(c) = V_{\delta_0}(c) \setminus \{c\}$ disjoint from A)

Thus $(*)$ is "vacuumally satisfied" for any positive $\delta \le \delta_0$ and hence $f(x)$ converges to <u>any</u> L as $x \to c$.

Case 2: $c \in A^c$

Th1. (Uniqueness and Sequential Criterion). Let $c \in A^c$ & $L \in \mathbb{R}$

(i) If $f(x) \to L$ and L' (with real L, L') as $x \to c$ then $L = L'$ (so notation $\lim_{x\to c} f(x) = L$ legitimate)

(ii) $f(x) \to L$ as $x \to c$ iff $[f(x_n) \to L$ whenever (x_n) is a seq in $A \setminus \{c\}$ convergent to $c]$

Proof (i). Let $\varepsilon > 0$. Then $\exists\, \delta, \delta' > 0$ s.t.

$$|f(x) - L| < \tfrac{\varepsilon}{2} \quad \forall\, x \in V_\delta(c) \cap (A \setminus \{c\})$$

$$|f(x) - L'| < \tfrac{\varepsilon}{2} \quad \forall\, x \in V_{\delta'}(c) \cap (A \setminus \{c\}).$$

Since $c \in A^c$, $\exists\, x \in V_{\delta \wedge \delta'}(c) \cap (A \setminus \{c\})$ and so, for this x, one has $|f(x) - L| < \frac{\varepsilon}{2}$ and $|f(x) - L'| < \varepsilon/2$ and consequently $|L - L'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, valid $\forall\, \varepsilon > 0$. Therefore we have $L = L'$.

(ii) "$\Rightarrow$" is left for you. To prove "$\Leftarrow$" part, we use the contradiction argument. Suppose $f(x) \not\to L$ as $x \to c$: $\exists\, \varepsilon > 0$ s.t. No $\delta > 0$ exists satisfying $(*)$; in particular, $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in V_{\frac{1}{n}}(c) \cap (A \setminus \{c\})$

s.t. $|f(x_n) - L| \geq \varepsilon$. Do this for all n, we have a seq (x_n) in $A \setminus \{c\}$ s.t. $0 \leq |x_n - c| < \frac{1}{n} \to 0$ & $|f(x_n) - L| \geq \varepsilon \ \forall n$ (so $f(x_n) \not\to L$ but $x_n \to c$ as $n \to \infty$) contradicting [....] in the statement Th 1 (ii).

· To avoid the triviality, let us assume from now on that c (or x_0) is in A^c, & $f : A \to \mathbb{R}$, $\ell, L \in \mathbb{R}$.

Th 2 (Boundness, Order-Preserving, Squeeze, cf Th 4.2.2).
Let $f, g : A \to \mathbb{R}$.

(i) If $\lim_{x \to c} f(x) = L \in \mathbb{R}$ then f is "locally bounded around c": $\exists\, \delta > 0$ and $M > 0$ such that

$|f(x)| \leq M \ \forall x \in V_\delta(c) \cap (A \setminus \{c\})$.

(ii) If $\alpha < \lim_{x \to c} f(x) = L < \beta$ then $\exists\, \delta > 0$ s.t.

$$\alpha < f(x) < \beta \quad \forall x \in V_\delta(c) \cap (A \setminus \{c\})$$

(ii*) If $\lim_{x \to c} f(x) = L \neq 0$ then $\exists\, \delta > 0$ s.t. $\frac{|L|}{2} \leq f(x) \leq \frac{3|L|}{2} \ \forall x \in V_\delta(c) \cap (A \setminus \{c\})$

(iii) If $f(x) \leq g(x) \ \forall x \in A \setminus \{c\}$, and $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = L'$ exist in $\mathbb{R}$ then $L \leq L'$.

(iv) If $f(x) \leq h(x) \leq g(x) \ \forall x \in A$ and $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L \in \mathbb{R}$ then $\lim_{x \to c} h(x) = L$.

proof. (i). Corresponding to $\varepsilon_0 := 1 > 0$, take $\delta > 0$ s.t. $|f(x) - L| < 1 \ \forall x \in V_\delta(c) \cap (A \setminus \{c\})$. Letting $M := |L| + 1$ one has $|f(x)| - L \leq |f(x) - L| < 1$ and so $|f(x)| < M \ \forall x \in V_\delta(c) \cap (A \setminus \{c\})$.

(ii). Let $\varepsilon = \min\{\beta - L, L-\alpha\}$ (>0). Pl. complete the argument.

(iii). Let $\varepsilon > 0$. Take $\delta, \delta' > 0$ s.t.

$$|f(x) - L| < \varepsilon/2 \quad \forall\, x \in V_\delta(c) \cap (A \setminus \{c\})$$

& $$|g(x) - L'| < \varepsilon/2 \quad \forall\, x \in V_{\delta'}(c) \cap (A \setminus \{c\}).$$

Since $c \in A^c$, take $x \in V_{\delta \wedge \delta'}(c) \cap (A \setminus \{c\})$.

Then $|f(x) - L| < \varepsilon/2$ & $|g(x) - L'| < \varepsilon/2$ and so

$$L - \tfrac{\varepsilon}{2} < f(x) \leq g(x) < L' + \varepsilon/2$$

implying that $L < L' + \varepsilon$, valid $\forall\, \varepsilon > 0$ so $L \leq L'$

(Note. You can also prove (iii) via (ii) by contradiction.)

(iv). Please do it yourself.

Th 3 (Computation Rules). Let $c \in A^c$ and $f, g : A \to \mathbb{R}$, $\lim_{x\to c} f(x) = l_1$ and $\lim_{x \to c} g(x) = l_2$ $(l_1, l_2 \in \mathbb{R})$. Then

(i) $\lim_{x \to c} (k f(x) + k' g(x)) = k l_1 + k' l_2 \quad \forall\, k, k' \in \mathbb{R}$.

(ii) $\lim_{x \to c} (f(x) g(x)) = l_1 l_2$

(iii) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}$ provided that $l_2, g(x) \neq 0 \ \forall x \in A$.

(iv) $\lim_{x \to c} |f(x)| = |l_1|$, $\lim_{x\to c} f^+(x) = l_1^+ \ (:= \max\{l_1, 0\})$, $\lim_{x \to c} f^-(x) = l_1^-$

(v) $\lim_{x \to c} (f \vee g)(x) = l_1 \vee l_2 \ (:= \max\{l_1, l_2\})$, $\lim_{x\to c} (f \wedge g)(x) = l_1 \wedge l_2$.

(vi) If $0 \leq f(x) \ \forall x \in A$ and $\lim_{x \to c} f(x) = L$ then $0 \leq L$ and $\lim_{x \to c} \sqrt{f(x)} = \sqrt{L}$.

Proof. We shall only do (ii) and (iii): the other parts for Ex.

(ii). Let $\varepsilon > 0$. Consider $\varepsilon_1, \varepsilon_2 > 0$ defined by

$$\varepsilon_1 := \min\left\{\frac{\varepsilon}{2M}, 1\right\}, \quad \varepsilon_2 = \min\left\{\frac{\varepsilon}{2M}, 1\right\}, \text{ where } M := |L| + |L'| + 1.$$

Since $\lim_{x\to c} f(x) = l_1$ + $\lim_{x\to c} g(x) = l_2$, $\exists\ \delta_1, \delta_2 > 0$ s.t.

$$|f(x) - l_1| < \varepsilon_1 \quad \forall\ x \in V_{\delta_1}(c) \cap (A \setminus \{c\}), \qquad (1)$$

$$|g(x) - l_2| < \varepsilon_2 \quad \forall\ x \in V_{\delta_2}(c) \cap (A \setminus \{c\}). \qquad (2)$$

Let $\delta = \delta_1 \wedge \delta_2$ (>0). Let $x \in V_\delta(c) \cap (A \setminus \{c\})$. Then

$$|f(x)| - |l_1| \leq |f(x) - l_1| < \varepsilon_1 \leq 1$$

$$|g(x)| - |l_2| \leq |g(x) - l_2| < \varepsilon_2 \leq 1$$

so $|f(x)| \leq |l_1| + 1 \leq M$ and $|g(x)| \leq |l_2| + 1 \leq M$ where $M := |l_1| + |l_2| + 1$.

Moreover

$$|f(x)g(x) - l_1 l_2| = |f(x)g(x) - l_1 g(x) + l_1 g(x) - l_1 l_2|$$

$$\leq |g(x)| \cdot |f(x) - l_1| + |l_1| \cdot |g(x) - l_2| \leq M \cdot |f(x) - l_1| + M|g(x) - l_2|$$

$$< M\varepsilon_1 + M\varepsilon_2 \leq M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon;$$

Therefore

$$|f(x)g(x) - l_1 l_2| < \varepsilon \quad \forall\ x \in V_\delta(c) \cap (A \setminus \{c\}),$$

proving (ii).

(iii). Let $\varepsilon > 0$. Let $\varepsilon_1, \varepsilon_2 > 0$ be defined by

$$\varepsilon_1 := \frac{m}{M}\left(\frac{\varepsilon}{2}\right)$$

$$\varepsilon_2 := \frac{m}{M}\left(\frac{\varepsilon}{2}\right)$$

where $M = |l_1| + |l_2|$ and $m = \frac{|l_2|^2}{2}$ (note that $M, m > 0$)

Take $\delta_1, \delta_2 > 0$ such that (1), (2) hold; and let $\delta := \min\{\delta_1, \delta_2\}$. It remains to check that

$$\left| \frac{f(x)}{g(x)} - \frac{l_1}{l_2} \right| < \varepsilon \quad \forall x \in V_\delta(x) \cap (A \setminus \{c\}) \quad (\#)$$

To do this, let $x \in V_\delta(c) \cap (A \setminus \{c\})$. Then, by (1) and (2), one has $|f(x) - l_1| < \varepsilon_1$ and $(-|g(x)| + |l_2|) \leq |g(x) - l_2| < \varepsilon_2 \leq \frac{|l_2|}{2}$ so $\frac{|l_2|}{2} \leq |g(x)|$

Consequently

$$\left|\frac{f(x)}{g(x)} - \frac{l_1}{l_2}\right| = \frac{|f(x)l_2 - g(x)l_1|}{|g(x)| \cdot |l_2|} = \frac{|f(x)\, l_2 - g(x)\, l_1 - l_1 l_2 + l_1 l_2|}{|g(x)| \cdot |\quad|} \leq$$

$$\leq \frac{|l_2| \cdot |f(x) - l_1| + |l_1| \cdot |l_2 - g(x)|}{|g(x)| \cdot |l_2|} \leq \frac{M\,|f(x) - l_1| + M\,|g(x) - l_2|}{m}$$

$$< \frac{M\varepsilon_1 + M\varepsilon_2}{m} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Note. Another method for proving Th 3 is via the sequential criterion together with the corresponding computation rules for sequences. Similarly the following examples can be seen in the same way, but we prefer to do by virtue of definitions.

Example 1. $\lim_{x \to 3} \frac{x^2+1}{x-2} = 10$ (Already done sequentially in --- by you)

Let $\varepsilon > 0$. Let $\delta > 0$ be defined by

$$\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{22}\right\}$$

Let $x \in V_\delta(3) \cap (A \setminus \{3\})$ where $A := \{x \in \mathbb{R} : x \neq 2\}$

It remains to show that $\left|\frac{x^2+1}{x-2} - 10\right| < \varepsilon$. To do this, note that $|x-3| < \delta \leq \frac{1}{2}$ so $2\frac{1}{2} < x < 3\frac{1}{2}$ & $\frac{1}{2} < |x-2|$ $1-|x-2| \leq |x-3| < \delta \leq \frac{1}{2}$

$$\left|\frac{x^2+1}{x-2} - 10\right| = \frac{|x^2-10x+21|}{|x-2|} = \frac{|(x-3)(x-7)|}{|x-2|} \leq 2|x-3| \cdot (4+7) < 22\delta \leq \varepsilon$$

Example 2 (We did this question for sequences) (so implicitly our function $x \to \frac{x^3+1}{x-2}$ is defined on $A := \{x \in \mathbb{R} : x \neq 2\}$)

$$\lim_{x \to 3} \frac{x^3+1}{x-2} = 28$$

Sol. Let $\varepsilon > 0$. Take $\delta > 0$ defined by

$$\delta := \min\left\{\frac{1}{2}, \frac{\varepsilon}{100}\right\}$$

(noting $V_{\frac{1}{2}}(3)$ is of distance $\geq 2\frac{1}{2} - 2 = \frac{1}{2}$ from 2)

Let $x \in V_\delta(3)$. Then $|x-3| < \delta \leq \frac{1}{2}$ so $|x| < 3\frac{1}{2} < 4$ & $1-|x-2| \leq |1-(x-2)| = |3-x| < \delta \leq \frac{1}{2}$ & $\frac{1}{2} \leq |x-2|$

(this can also be reached in the following way: $3-\frac{1}{2} < x < 3+\frac{1}{2}$ so $|x| < 3\frac{1}{2} < 4$ and $\frac{1}{2} < x-2 = |x-2|$). Consequently

$$\left|\frac{x^3+1}{x-2} - 28\right| = \left|\frac{x^3-28x+57}{x-2}\right| = \frac{|x-3| \cdot |x^2+3x-19|}{|x-2|}$$

$$\leq \frac{|x-3| \cdot (|x|^2+3|x|+19)}{|x-2|} \leq \frac{|x-3|(4^2+3\times 4+19)}{1/2} \leq \frac{47|x-3|}{1/2}$$

$$\leq 100|x-3| < 100\delta \leq \varepsilon,$$ valid for all $x \in V_\delta(3)$

The following result concerns:

$$T \xrightarrow{g} A \xrightarrow{f} \mathbb{R} \quad (T, A \subseteq \mathbb{R})$$

True or not (??) that

$$\lim_{t \to t_0} f(g(t)) = \lim_{x \to \lim_{t \to t_0} g(t)} f(x) \qquad (??)$$

if $l := \lim_{x \to x_0} f(x)$ and $x_0 := \lim_{t \to t_0} g(t)$ exist in $\mathbb{R}$

(so-called Chain-Rule). The answer is "no" in general, e.g. g is a constant function (say $g(t) = x_0 \ \forall t$ so $\lim_{t \to t_0} g(t) = x_0 \ \forall t_0$) while g is "singular at x_0":

$$f = \chi_{\{x_0\}}, \text{ i.e. } f(x) = \begin{cases} 0 & \text{if } x \neq x_0 \\ 1 & \text{if } x = x_0 \end{cases}$$

so $\lim_{x \to x_0} f(x) = 0$ while

$$f(g(t)) = f(x_0) = 1 \qquad \left(\text{so } \lim_{x \to x_0} f(g(x)) = 1 \ \forall x_0 \right)$$

Thus LHS of (??) $= 1$

RHS of (??) $= 0$.

The difficulty of the above example lies in the fact that

$$\lim_{x \to x_0} f(x)$$

has nothing to do with the value of f at x_0 but on its near-by points because in of the definition for $l := \lim_{x \to x_0} f(x)$ on concerns

$$V_\delta(x_0) \cap (A \setminus \{x_0\}) = V_\delta(x_0) \setminus \{x_0\}$$

(our present $A = \mathbb{R}$).

punched neighbourhood

Th 4 (Chain Rule). Let $f : A \to \mathbb{R}$, $x_0 \in A^c$ and $l \in \mathbb{R}$ be s.t.

$$l := \lim_{x \to x_0} f(x)$$

Let $g : T \to A$, $t_0 \in T^c$ be s.t.

$$x_0 = \lim_{t \to t_0} g(t)$$

Suppose $\exists\ \gamma_0 > 0$ s.t

$$g(t) \neq x_0 \quad \forall\, t \in V_{\gamma_0}(t_0) \cap (T \setminus \{t_0\})$$

Then

$$\lim_{t \to t_0} f(g(t)) = l$$

Proof. Let $\varepsilon > 0$. Since $l = \lim_{x \to x_0} f(x)$, $\exists\, \delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - x_0| < \delta \ \& \ x \in A \quad (1)$$

Since $\lim_{t \to t_0} g(t) = x_0$, $\exists\, \gamma > 0$ s.t.

$$|g(t) - x_0| < \delta \text{ whenever } t \in V_\gamma(t_0) \cap (T \setminus \{t_0\})$$

By the given property of γ_0, it follows that

$$0 < |g(t) - x_0| < \delta \text{ whenever } t \in V_{\gamma \wedge \gamma_0}(t_0) \cap (T \setminus \{t_0\})$$

and so, by (1),

$$|f(g(t)) - l| < \varepsilon \text{ whenever } t \in V_{\gamma \wedge \gamma_0}(t_0) \cap (T \setminus \{t_0\})$$

This shows that $\lim_{t \to t_0} f(g(t)) = l$.

Extensions on Limit Concepts

(one-sided limits, limit at infinity, infinite limit).

1. With our implicit understanding ($f : A \to \mathbb{R}$ and $c \in A^c$) our notation

$\lim_{x \to c} f(x)$ should be more precisely denoted as

$$\lim_{A \ni x \to c} f(x).$$

For the same, the standard notation for left-side limit

$\lim_{x \to c^-} f(x)$ is meant to be

$$\lim_{\substack{x \to c \\ x \in A \cap (-\infty, c)}} f(x) \qquad \left(\begin{array}{c} \text{assuming that} \\ c \in \left(A \cap (-\infty, c)\right)^c : \\ V_\delta(c) \cap \left(A \cap (-\infty, c)\right) \neq \emptyset \\ \forall \delta > 0 \end{array} \right).$$

Thus

$$l := \lim_{x \to c^-} f(x) \quad (\in \mathbb{R})$$

means, by definition that $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad x \in V_\delta(c) \cap \left(A \cap (-\infty, c)\right),$$

i.e.

$$|f(x) - l| < \varepsilon \quad \text{whenever } x \in A \text{ and } c - \delta < x < c.$$

You are ask to give the definition for $\lim_{x \to c+} f(x)$.

Exercises. (1) Negation for each of these new concepts

(ii) Establish the sequential criterion.

(two methods : (a) apply old result to $A \cap (-\infty, c)$ in place of A

(b) model the old proof.

II. $\lim_{x \to +\infty} f(x)$ (Assuming that $\forall \delta > 0 \ \exists \ a \in A$ s.t. $a \geq \delta$)

$\lim_{x \to -\infty} f(x)$

III. $\lim_{x \to c} f(x) = \begin{cases} +\infty \\ -\infty \\ l \in \mathbb{R} \end{cases}$

& also that when "$x \to c$" is replaced by

"$x \to c^-$" or

"$x \to c+$" or

"$x \to +\infty$" or

"$x \to -\infty$"

All have sequential criterion results

The following result is evident

Th 5. Suppose $c \in A_+^c \cap A_-^c$ where $A_+ := A \cap (c, +\infty)$ etc.

Then

$\lim_{x \to c} f(x)$ exists in $[-\infty, \infty]$ iff $\lim_{x \to c-} f(x) = \lim_{x \to c^+} f(x) \in [-\infty, \infty]$.

Warning:

1. Regarding Cauchy criterion : not true for $\lim_{x \to c} f(x) = \begin{cases} +\infty \\ -\infty \end{cases}$

but true for $\lim_{x \to c} f(x)$ exists in $\mathbb{R}$: $\forall \varepsilon > 0 \ \exists$

$\delta > 0$ such that

$$|f(x)-f(u)| < \varepsilon \text{ whenever } u, x \in V_\delta(c) \cap (A \setminus \{c\})$$

2. Regarding product/quotient rules: not true for "$\frac{0}{0}$" or "$0 \cdot \infty$" or "$\frac{\infty}{\infty}$" types.

e.g. $f(x) = x^2$

$g_1(x) = x$; $g_2(x) = x^2$; $g_3(x) = x^3$; $g_4(x) = 4x^2$

$g_5(x) = 5x^2$ etc.

Note that $\lim_{x\to c} |f(x)| = \infty$ iff $\lim_{x\to c} \frac{1}{f(x)} = 0$ (provided that $f(x) \neq 0 \; \forall x$)

Examples (cf. p118, Q5(b)).

1. $\lim_{x\to 1^+} \frac{x}{x-1} = +\infty$

2. $\lim_{x\to 1^-} \frac{x}{x-1} = -\infty$.

3. $\lim_{x\to 1} \frac{x}{x-1}$ not exist.

Analysis for Q1.

$$\frac{x}{x-1} > n$$

iff $\frac{x-1+1}{x-1} > n$

iff $1 + \frac{1}{x-1} > n$

iff $\frac{1}{x-1} > n-1$ (Assuming $n \geq 2$ and $x > 1$)

iff $\frac{1}{n-1} > x-1 > 0$

iff $1+\frac{1}{n-1} > x > 1$

Sol of Q1. Let $n \in \mathbb{N}$ and $n \geq 2$. Take $\delta := \frac{1}{n-1}$. Then, $\forall x \in V_\delta(1) \cap (1,+\infty)$ (i.e. $1 < x < 1+\delta$) one has

$$0 < x-1 < \frac{1}{n-1} \quad \text{so} \quad n < 1 + \frac{1}{x-1} = \frac{x}{x-1}$$

Sol. of Q2. Let $n \in \mathbb{N}$. Take $\delta := \frac{1}{n+1}$. Then,

if $1-\delta < x < 1$ then $0 < 1-x < \delta$ so $\frac{1}{1-x} > n+1$

& $-n > 1 - \frac{1}{1-x} = \frac{-x}{1-x} = \frac{x}{x-1}$

The above proof of Q2 is motivated from the following:

For $x<1$,

$$\frac{x}{x-1} < -n$$

$$\Longleftrightarrow \frac{x-1+1}{x-1} < -n$$

$$\Longleftrightarrow 1 + \frac{1}{x-1} < -n$$

$$\Longleftrightarrow 1+n < \frac{1}{1-x} \Longleftrightarrow 1-x < \underbrace{\frac{1}{1+n}}_{\text{def } \delta}$$

Sol. of Q3. By Th 5 and Q 1, Q2.

Also, you can try to do in the following way (Without Q 1 & Q2 nor Th 5)

each of the three cases would lead contradiction

$$\lim_{x\to 1} \frac{x}{x-1} = \begin{cases} l \in \mathbb{R} \\ +\infty \\ -\infty \end{cases}$$

Ex. Comparison Test etc. $f(x) \le g(x) \ \forall x \in \mathbb{R}$

1. $\lim\limits_{\substack{x\to c \\ -\infty \\ +\infty}} f(x) = +\infty \Rightarrow \lim g(x) = +\infty$

2. $\lim g(x) = -\infty \Rightarrow \lim f(x) = -\infty$

Th 6 (Cauchy Criterion). Let $f : X \to \mathbb{R}$ and $c \in B^c$ $\left(\begin{array}{l} \text{e.g. } X = A \cap (c, \infty) \text{ or} \\ X = A \\ X = A \cap (-\infty, c) \end{array}\right)$

Then (i) $\iff$ (ii) where

(i) $\exists\, l \in \mathbb{R}$ s.t $\lim_{X \ni x \to c} f(x) = l$, i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

(*) $|f(x) - l| < \varepsilon \quad \forall x \in V_\delta(c) \cap (X \setminus \{c\})$

(ii) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

(**) $|f(u) - f(x)| < \varepsilon \quad \forall x, u \in V_\delta(c) \cap (X \setminus \{c\})$.

Th 6*. Let $f : [a, \infty) \to \mathbb{R}$. Then

(i) $\lim_{x \to +\infty} f(x) = l$ for some $l \in \mathbb{R}$

(ii) $\forall \varepsilon > 0\ \exists M > a$ s.t.

$|f(u) - f(x)| < \varepsilon \quad \forall u, x \in [M, +\infty)$

Proof (ii) $\Rightarrow$ (i). Let (x_n) be a seq in $[a, \infty)$ s.t $\lim_n x_n = +\infty$ (e.g. $(x_n) = (n)$, starting with $N \geq a$). Then (ii), $(f(x_n))$ is a Cauchy seq. Indeed, $\forall \varepsilon > 0$, take M correspondingly as in (ii).

Since $\lim_n x_n = +\infty$, $\exists\, N \in \mathbb{N}$ s.t.

$$x_n \geqslant M \quad \forall\, n \geqslant N$$

and it follows from the displayed line in (ii) that

$$|f(x_m) - f(x_n)| < \varepsilon \quad \forall\, m, n \geqslant N.$$

Therefore $(f(x_n))$ is Cauchy and hence converges, say to $\ell \in \mathbb{R}$. It remains to show that

$$\lim_{x \to +\infty} f(x) = \ell.$$

To do this, let $\varepsilon > 0$. Then, $\exists\, N_1 \in \mathbb{N}$ s.t.

$$① \quad |f(x_n) - \ell| < \varepsilon/2 \quad \forall\, n \geqslant N_1$$

and $\exists\, M > a$ s.t.

$$② \quad |f(u) - f(x)| < \varepsilon/2 \;\; \forall\, u, x \geqslant M.$$

Let $N \geqslant N_1, M$. Then

$$|f(x_N) - \ell| < \varepsilon/2$$

&

$$|f(u) - f(x_N)| < \varepsilon/2 \quad \forall\, u \geqslant M$$

and so
$$|f(u)-\ell| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \quad \forall\ u \geq M$$
showing that $\lim_{u\to+\infty} f(u) = \ell$.

(i) $\Rightarrow$ (ii). Easier.

Th 6 is proved similarly.

Ex1. Let $p(x) = ax^{2n+1} + a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x^1 + a_0 \ \forall x$ where $a>0$ and $a_i \in \mathbb{R}\ \forall i$. Then $\lim_{x\to+\infty} p(x) = +\infty$ (and $\lim_{x\to-\infty} p(x) = -\infty$)

Sol. $\lim_{x\to+\infty} \frac{p(x)}{x^{2n+1}} = a > 0$ so $\exists\ M>0$ s.t.
$$2a > \frac{p(x)}{x^{2n+1}} > \frac{a}{2} \qquad \forall\ x \geq M$$
so
$$p(x) > \frac{a}{2}\left(x^{2n+1}\right) \to +\infty \text{ as } x\to+\infty.$$
(we assume the basic alg. property leading to $\lim_{x\to\infty} x^{2n+1} = +\infty$!
$$x^{2n+1} - M^{2n+1} = (x-M)\left(x^{2n} + x^{2n-1}M + x^{2n-2}M^2 + \cdots + xM^{2n-1} + M^{2n}\right))$$

Ex. Let $q(x) = ax^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x + a_0 \ \forall x$ and $a \neq 0$. Then $\lim_{|x|\to\infty} q(x) = +\infty$.